Shape Sensitivity Analysis of Piezoelectric Structures by the Adjoint Variable Method

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Shape sensitivity expressions are derived for linear piezoelectric structures with coupled mechanical and elastic fields. By adopting the quasielectrostatic approximation for these inherently anisotropic materials, the adjoint variable method of optimization and the material derivative formulation of shape variations are used in a systematic procedure to evaluate the total variation of a general performance criterion with respect to shape variations. The material (total) derivative of the adopted integral functional is found in terms of primary and adjoint quantities, as well as the deformation velocity field. Since the structure is assumed to undergo dynamic response, domain integrations evaluated at the initial time are also needed in this formulation.

Introduction

THERE has been a constantly growing interest in the use of piezoelectric crystals for active control of space structures. For example, the vibratory response of large flexible structures can be controlled by using piezoelectric sensors and actuators. In this respect, flexural vibrations of beams have been optimally controlled by using such elements by Hanagud et al.,[1] Gerhold and Rocha,[2] and others. The direct effect in piezoelectric structures may be defined as electric polarization produced by mechanical strain in crystals belonging to certain classes, the polarization being proportional to the strain and changing sign with it (see, e.g., Refs. 3 and 4). In the converse (i.e., reciprocal, or inverse) effect, a piezoelectric crystal becomes strained when electrically polarized by an amount proportional to the polarizing field. Besides being employed as active control elements, piezoelectric structures may also be used as integral parts of various engineering systems, with their basic character of coupled response to electric and mechanical fields. In all such applications, the determination of optimal shapes of such structures is of much importance, especially in space applications where weight restrictions are stringent.

For an effective optimal shape design, shape sensitivity analysis of piezoelectric structures must usually be performed. By deriving sensitivity expressions of the system's response with respect to shape configuration variations, it is possible to use gradient-based minimization techniques in the numerical solution of shape optimization problems. In the present paper, the adjoint variable method (AVM) of optimization will be used to derive shape sensitivity expressions for linear piezoelectric structures. By adopting a continuum approach, the material (i.e., total) variations of integrals with respect to shape variations will be performed by using the material derivative concept of structural optimization. Shape sensitivity analyses of coupled systems have been given in the literature by using similar procedures for thermoelastic structures by Meric.[5-9] After the derivation of the sensitivity expressions for a general performance criterion, an analytical example problem will be provided to check the validity of the results.

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Problem Formulation

Primary Problem

In the Voigt's linear theory of piezoelectricity (see, e.g., Maugin[4] and Holland and Eer Nisse[10]), the field equations may be given within a quasielectrostatic approximation, as follows:

$$\text{in } V: \quad T_{ij,j} + F_i = \rho \ddot{u}_i \tag{1}$$

$$D_{m,m} = O \tag{2}$$

where, by adopting the standard notation on piezoelectric crystals,[11] T_{ij} refers to the elastic stress tensor, F_i is the body force vector, ρ is the density, u_i indicates elastic displacements, the superpositioned dot refers to time differentiation, D_m is the electric displacement vector, and V is the volume of the structure. It has been assumed in Eq. (2) that there are no free charges present in the structure. The linear constitutive equations pertinent to the piezoelectric structure can be written in the following form:

$$T_{ij} = C_{ijkl} u_{k,l} + e_{nij} \phi_{,n} = T_{ji} \tag{3}$$

$$D_m = e_{mkl} u_{k,l} - \epsilon_{mn} \phi_{,n} \tag{4}$$

where C_{ijkl} is the elastic stiffness tensor, e_{nij} is the piezoelectric tensor, ϕ indicates the electric potential whose gradient gives the negative of the electric field, and ϵ_{mn} indicates the dielectric tensor. It is noted that all fields are assumed to be small (in a certain sense), and therefore, the constitutive equations are not sufficient when extremely intense electric fields are present. It is known that piezoelectric substances are inherently anisotropic, i.e., no piezoelectric body can have a center of symmetry, or, in particular, be isotropic. Of the 32 crystal classes, only 20 allow piezoelectricity.[3] Among the piezoelectric substances, Rochelle salt, lithium sulfate, and quartz are prominent as transducers, while various types of ceramic compounds, such as lead zirconate barium titanate (designated as G-1195) or polymeric piezoelectric polyvinylidence fluoride (PVDF), are utilized as vibration controllers. The material properties that are taken as homogeneous in the present paper may enjoy certain symmetry properties, which are summarized as follows:

$$C_{ijkl} = C_{klij}, \qquad e_{nij} = e_{nji}, \qquad \epsilon_{mn} = \epsilon_{nm}$$

Thus, C_{ijkl}, e_{nij}, and ϵ_{mn} tensors may have at most 21, 18, and 6 independent coefficients, respectively.

Appropriate boundary conditions must be given at all points on the boundary surface S of volume V. Mechanically, displacements and tractions may be prescribed over S_u and S_T portions of the boundary, respectively; i.e.,

$$\text{on } S_u: \quad u_i = u_i^0 \tag{5}$$

$$\text{on } S_T: \quad T_i = T_{ij} n_j = T_i^0 \tag{6}$$

where $S = S_u U S_T$, the superscript zero refers to prescribed values, T_i is the traction vector, and n_i is the unit vector normal to S. Electrically, one may specify the electric potential or the free surface charge density over portions of the boundary as follows:

$$\text{on } S_\phi: \quad \phi = \phi^0 \tag{7}$$

$$\text{on } S_D: \quad D = D_i n_i = D^0 \tag{8}$$

where $S = S_\phi U S_D$, and D is the normal component of the electric displacement vector.

In the quasielectrostatic approximation adopted in the present analysis initial conditions are also specified for the mechanical field in the following form:

$$\text{at } t = O: \quad u_i = u_{i0} \tag{9}$$

$$\dot{u}_i = \dot{u}_{i0} \tag{10}$$

where t is the time variable and the subscript zero refers to prescribed initial values. Equations (1-10) thus constitute the governing equations of the so-called primary problem for a linear piezoelectric structure.

General Performance Criterion

It is noted that if all the material properties, distributed and boundary "loads," initial conditions, and the shape configurations of the structure are given a priori, the unknown variables of the problem, i.e., displacements, stresses, potential and electric potential values, etc., may be solved directly. If, on the other hand, any one of the aforementioned data is missing explicitly, the problem may be termed as an inverse problem. In particular, a shape optimization problem is a shape inverse problem in which the shape configuration of the structure is optimally found so as to satisfy some physical requirements. Such physical constraints may be cast mathematically into integral functional forms representing the structure's response to loads for an assumed shape configuration. As such, a general performance criterion I is now considered in the following form:

$$I = \int_\theta \int_V f(u_i, T_{ij}, \phi, D_i)\, dV\, dt + \int_\theta \int_S g(u_i, T_i, \phi, D)\, dS\, dt + \int_V [h(u_i)]_\theta\, dV \tag{11}$$

where $\theta = [O, t_f]$ represents the period of time of "observation" of the dynamic structure, t_f is the final time, and the symbol $[\cdot]_\theta$ indicates

$$[\cdot]_\theta = [\cdot]_{t=t_f} - [\cdot]_{t=0}$$

The functions f, g, and h are assumed to be continuously differentiable functions with respect to their arguments.

The performance criterion I may represent an objective function to be minimized or a behavioral constraint to be satisfied. The aim of the present shape sensitivity analysis (SSA) is to evaluate the total change in I as the structural shape is varied. It is noted that the functional I depends on the shape of the piezoelectric structure for two reasons: firstly, the integrands are implicit functions of shape, and secondly, the integrations are performed over the volume and boundary surface of the structure.

Shape Sensitivity Analysis

The SSA of the general performance criterion I [Eq. (1)] for the piezoelectric structure governed by Eqs. (1-10) will be performed by using the adjoint variable method (AVM) and the material derivative (MD) concept. In this procedure (see, e.g., Ref. 9), the field equations (1) and (2) are first incorporated into the functional I in a weighted residual sense by introducing adjoint (i.e., Lagrange multiplier) variables. Denoting the adjoint elastic displacements by u_i^* and adjoint electric potential by ϕ^*, an augmented functional $\hat{I}$ can thus be defined as

$$\hat{I} = \int_\theta \int_V [f + u_i^*(T_{ij,j} + F_i - \rho \ddot{u}_i) + \phi^* D_{m,m}]\, dV\, dt + \int_\theta \int_S g\, dS\, dt + \int_V [h]_\theta\, dV \tag{12}$$

which, after integration by parts, yields the following:

$$\hat{I} = \int_\theta \int_V [f - T_{ij} u_{i,j}^* + F_i u_i^* + \rho \dot{u}_i \dot{u}_i^* - D_m \phi_{,m}^*]\, dV\, dt + \int_\theta \int_S (g + T_i u_i^* + D\overset{*}{\phi})\, dS\, dt + \int_V [h - \rho \dot{u}_i u_i^*]_\theta\, dV \tag{13}$$

Material Derivatives

The problem of any SSA is to compute explicitly the derivatives of performance criteria with respect to the decision variables. In the present study, the structural shape itself represents the decision variable. The SSA of elastic structures has been investigated by the material derivative (MD) concept of continuum mechanics in a book by Haug et al.[12] The same technique has been employed by the present authors for coupled systems as well (see, e.g., Refs. 6-9). In the MD formulation of shape variations, the structural volume V is treated as a continuous medium undergoing a "dynamic" deformation, characterized by a pseudotime parameter τ, as V is transformed under a transformation in the SSA. Thus, a point x_i in V moves to the point x_i^τ in the varied domain V^τ, given by

$$x_i^\tau = x_i + \tau V_i \tag{14}$$

where the so-called deformation velocity $V_i = V_i(x_j)$ is defined in the whole space. The MD of a continuously differentiable function w, which satisfies a boundary value problem defined in V, can be given by

$$\partial_M w = w' + w_{,k} V_k \tag{15}$$

where $\partial_M w$ and w' represent the MD and the partial derivative of w (with respect to τ), respectively. It is noted that the preceding formula is very similar to the total (i.e., material) derivative formula for continuous functions in continuum mechanics, where the total differentiation is with respect to the (real) time variable t. By following Meric (1988b and c), the material and partial derivatives of a space-time dependent function w are given as

$$\partial_M w(x_i, t) = \lim_{\tau \to 0} \frac{w^\tau(x_i + \tau V_i, t) - w(x_i, t)}{\tau}$$

$$w'(x_i, t) = \lim_{\tau \to 0} \frac{w^\tau(x_i, t) - w(x_i, t)}{\tau}$$

The partial derivatives with respect to x_i and τ commute with each other.

Since the augmented functional $\hat{I}$ [Eq. (13)] involves volume-time and space-time integrals, the MD of these integrals are needed in the SSA. General MD formulas may now be given for such integrals in the following form:

The MD of a general volume-time integral defined by

$$\Psi = \int_\theta \int_V w \, dV \, dt \tag{16}$$

is given by

$$\partial_M \Psi = \int_\theta \int_V w' \, dV \, dt + \int_\theta \int_V w V_n \, dS \, dt \tag{17}$$

where V_n is the normal component of V_i, and it has been assumed that there exist no discontinuities of w within V.

The MD of a general surface-time integral defined by

$$\Psi = \int_\theta \int_V w \, dS \, dt \tag{18}$$

is, on the other hand, given by

$$\partial_M \Psi = \int_\theta \int_V [w' + (w_{,n} + Hw)V_n] \, dS \, dt + \int_\theta \oint_\Gamma [\![w]\!] V_\mu \, d\Gamma \, dt \tag{19}$$

where H is the curvature of the boundary S in R^2 and twice the mean surface curvature of S in R^3, and $(\cdot)_{,n}$ denotes the normal derivative of $(\cdot)$ on S. Any discontinuity of w across the boundary surface curve $\Gamma \epsilon S$, indicated by the notation $[\![w]\!] = w^- - w^+$, where w^- and w^+ are the values of w at the negative and positive sides of Γ, is taken care of by the last integral over $\theta \times \Gamma$ in Eq. (19), where V_μ is the component of V_i on S, normal to Γ and tangent to S.

By employing Eqs. (16–19), the MD of $\hat{I}$ can be taken as follows:

$$\begin{aligned}
\partial_M \hat{I} = & \int_\theta \int_V \bigg[\frac{\partial f}{\partial u_i} u_i' + \left(\frac{\partial f}{\partial T_{ij}} - u_{i,j}^* \right) T_{ij}' + \frac{\partial f}{\partial \phi} \phi' \\
& + \left(\frac{\partial f}{\partial D_m} - \phi_{,m}^* \right) D_m' - T_{ij} u_{i,j}^{*\prime} + F_i u_i^{*\prime} + u_i^* F_i' \\
& + \rho \dot{u}_i \dot{u}_i^{*\prime} + \rho \dot{u}_i^* \dot{u}_i' - D_m \phi_{,m}^{*\prime} \bigg] \, dV \, dt \\
& + \int_\theta \int_S [f - T_{ij} u_{i,j}^* + F_i u_i^* + \rho \dot{u}_i \dot{u}_i^* - D_m \phi_{,m}^*] \\
& + [(g + T_i u_i^* + D\phi^*)_{,n} + H(g + T_i u_i^* + D\phi^*)] V_n \, dS \, dt \\
& + \int_\theta \int_S \bigg[\frac{\partial g}{\partial u_i} u_i' + \left(\frac{\partial g}{\partial T_i} + u_i^* \right) T_i' + \frac{\partial g}{\partial \phi} \phi' \\
& + \left(\frac{\partial g}{\partial D} + \phi^* \right) D' + T_i u_i^{*\prime} + D \phi^{*\prime} \bigg] \, dS \, dt \\
& + \int_\theta \oint_\Gamma [\![g + T_i u_i^* + D\phi^*]\!] \, V_\mu \, d\Gamma \, dt \\
& + \int_V \left[\frac{\partial h}{\partial u_i} u_i' - \rho \dot{u}_i u_i^{*\prime} - \rho u_i^* u_i' \right]_\theta dV \\
& + \int_S [h - \rho \dot{u}_i u_i^*]_\theta V_n \, dS
\end{aligned} \tag{20}$$

The partial derivative (with respect to τ) of the constitutive Eqs. (3) and (4) is simply given by

$$T_{ij}' = C_{ijkl} u_{k,l}' + e_{nij} \phi_{,n}' \tag{21}$$

$$D_m' = e_{mkl} u_{k,l}' - \epsilon_{mn} \phi_{,n}' \tag{22}$$

These equations are introduced into $\partial_M \hat{I}$ [Eq. (20)] and now used to increase the order of differentiations. In particular, a term designated by π_1' and defined by

$$\begin{aligned}
\pi_1' &= \int_\theta \int_V \left(\frac{\partial f}{\partial T_{ij}} - u_{i,j}^* \right) T_{ij}' \, dV \, dt \\
&= \int_\theta \int_V \left(\frac{\partial f}{\partial T_{ij}} - u_{i,j}^* \right) (C_{ijkl} u_{k,l}' + e_{nij} \phi_{,n}') \, dV \, dt
\end{aligned}$$

yields after integration by parts the following

$$\begin{aligned}
\pi_1' = & \int_\theta \int_V \left(\frac{\partial f}{\partial T_{ij}} - u_{i,j}^* \right) (C_{ijkl} u_k' n_l + e_{nij} \phi' n_n) \, dS \, dt \\
& - \int_\theta \int_V \left[\left(\frac{\partial f}{\partial T_{ij}} - u_{i,j}^* \right)_{,l} C_{ijkl} u_k' + \left(\frac{\partial f}{\partial T_{ij}} - u_{i,j}^* \right)_{,n} e_{nij} \phi' \right] dV \, dt
\end{aligned}$$

which is then transformed into

$$\begin{aligned}
\pi_1' = & \int_\theta \int_S \left(\frac{\partial f}{\partial T_{kl}} - u_{k,l}^* \right) (C_{ijkl} n_j u_i' + e_{jlk} n_j \phi') \, dS \, dt \\
& - \int_\theta \int_V \left[\left(\frac{\partial f}{\partial T_{kl}} - u_{k,l}^* \right)_{,j} (C_{ijkl} u_i' + e_{jlk} \phi') \right] dV \, dt
\end{aligned} \tag{23}$$

by using the symmetry properties of the C_{ijkl} and e_{ijk} tensors.

A term in $\partial_M \hat{I}$, denoted by π_2' and defined by

$$\pi_2' = \int_\theta \int_V \left(\frac{\partial f}{\partial D_m} - \phi_{,m}^* \right) D_m' \, dV \, dt$$

similarly yields the following expression

$$\begin{aligned}
\pi_2' = & \int_\theta \int_S \bigg[\left(\frac{\partial f}{\partial D_k} - \phi_{,k}^* \right) e_{kji} n_j u_i' \\
& - \left(\frac{\partial f}{\partial D_n} - \phi_{,n}^* \right) \epsilon_{mn} n_m \phi' \bigg] \, dS \, dt \\
& - \int_\theta \int_V \bigg[\left(\frac{\partial f}{\partial D_k} - \phi_{,k}^* \right)_{,j} e_{kji} u_i' \\
& - \left(\frac{\partial f}{\partial D_n} - \phi_{,n}^* \right)_{,m} \epsilon_{mn} \phi' \bigg] \, dV \, dt
\end{aligned} \tag{24}$$

using Eq. (22), integrations by parts, and the symmetry properties of the material tensors.

After all integrations by parts are performed, the MD of the augmented functional can be written as follows:

$$\begin{aligned}
\partial_M \hat{I} = & \int_\theta \int_V \bigg[\left(T_{ij,j}^* + \frac{\partial f}{\partial u_i} - \rho \ddot{u}_i^* \right) u_i' + \left(D_{m,m}^* + \frac{\partial f}{\partial \phi} \right) \phi' \\
& + (T_{ij,j} + F_i - \rho \ddot{u}_i) u_i^{*\prime} + D_{m,m} \phi^{*\prime} + u_i^* F_i' \bigg] \, dV \, dt \\
& + \int_\theta \int_S [f - T_{ij} u_{i,j}^* + F_i u_i^* + \rho \dot{u}_i \dot{u}_i^* - D_m \phi_{,m}^* \\
& + (g + T_i u_i^* + D\phi^*)_{,n} + H(g + T_i u_i^* + D\phi^*)] V_n \, dS \, dt \\
& + \int_\theta \int_S \bigg[\left(\frac{\partial g}{\partial u_i} - T_i^* \right) u_i' + \left(\frac{\partial g}{\partial T_i} + u_i^* \right) T_i' \\
& + \left(\frac{\partial g}{\partial \phi} - D^* \right) \phi' + \left(\frac{\partial g}{\partial D} + \phi^* \right) D' \bigg] \, dS \, dt \\
& + \int_\theta \oint_\Gamma [\![g + T_i u_i^* + D\phi^*]\!] \, V_\mu \, d\Gamma \, dt
\end{aligned}$$

$$+\int_V\left[\left(\frac{\partial h}{\partial u_i}+\rho\dot{u}_i^*\right)u_i'-\rho u_i^*\dot{u}_i'\right]_\theta \mathrm{d}V$$

$$+\int_S[h-\rho\dot{u}_i u_i^*]_\theta V_n\,\mathrm{d}S \tag{25}$$

where the adjoint stress tensor T_{ij}^* and the adjoint electric displacement vector D_m^* are given by the following adjoint constitutive equations:

$$T_{ij}^*=C_{ijkl}\left(u_{k,l}^*-\frac{\partial f}{\partial T_{kl}}\right)+e_{nij}\left(\phi^*,_n-\frac{\partial f}{\partial D_n}\right) \tag{26}$$

$$D_m^*=e_{mkl}\left(u_{k,l}^*-\frac{\partial f}{\partial T_{kl}}\right)-\epsilon_{mn}\left(\phi^*,_n-\frac{\partial f}{\partial D_n}\right) \tag{27}$$

The boundary conditions (5–8) can be introduced into Eq. (25) by using the MD formula for continuous functions [Eq. (15)]. For example, taking the MD of the displacement boundary condition (5) and rearranging the terms, it can be shown that

$$\text{on } S_u:\quad u_i'=\partial_M u_i^0-u_{i,k}V_k \tag{28}$$

where $\partial_M u_i^0$ is known from the MD form of the prescribed displacements on S_u. Similar expressions follow for the other type of boundary conditions, which are stated as follows:

$$\text{on } S_T:\quad T_i'=\partial_M T_i^0-T_{i,k}V_k \tag{29}$$

$$\text{on } S_\phi:\quad \phi'=\partial_M\phi^0-\phi,_kV_k \tag{30}$$

$$\text{on } S_D:\quad D'=\partial_M D^0-\mathrm{D},_kV_k \tag{31}$$

The initial conditions prescribed on the mechanical state (or primary) variables [Eqs. (9) and (10)], can also be introduced into Eq. (15). It, thus, follows that

$$\text{at } t=0:\quad u_i'=\partial_M u_{i0}-u_{i,k}V_k \tag{32}$$

$$\dot{u}_i'=\partial_M\dot{u}_{i0}-\dot{u}_{i,k}V_k \tag{33}$$

where the MD forms of the initial displacements and velocities are again known from their respective prescribed distributions.

Adjoint Problem

As stated previously, the aim of the present SSA is to find the sensitivity of the general performance criterion I with respect to shape variations; in particular, explicitly in terms of the deformation velocity field V_i. Hence, the partial derivative terms (with respect to τ) of the primary variables present in Eq. (25) must be eliminated. By equating the coefficients of such unwanted terms equal to zero, an adjoint problem may be defined in the following form:

$$\text{in } V:\quad T_{ij,j}^*+\frac{\partial f}{\partial u_i}=\rho\ddot{u}_i^* \tag{34}$$

$$D_{m,m}^*+\frac{\partial f}{\partial\phi}=0 \tag{35}$$

$$\text{on } S_u:\quad u_i^*=-\frac{\partial g}{\partial T_i} \tag{36}$$

$$\text{on } S_T:\quad T_i^*=T_{ij}^*n_j=\frac{\partial g}{\partial u_i} \tag{37}$$

$$\text{on } S_\phi:\quad \phi^*=-\frac{\partial g}{\partial D} \tag{38}$$

$$\text{on } S_D:\quad D=\frac{\partial g}{\partial\phi} \tag{39}$$

$$\text{at } t=t_f:\quad u_i^*=0 \tag{40}$$

$$\dot{u}_i^*=-\frac{1}{\rho}\frac{\partial h}{\partial u_i} \tag{41}$$

It is noted that the adjoint problem governed by Eqs. (34–41) is characterized as a boundary-value final-time problem. The adjoint constitutive equations have been given previously by Eqs. (26) and (27). The mathematical structure of the adjoint problem is seen to be very similar to that of the primary problem, except for the fact that final-time conditions are prescribed at $t=t_f$, and that the partial differential equations must be solved backwards in time from $t=t_f$ to $t=0$. The adjoint problem governed by Eqs. (34–41) corresponds to the general performance criterion originally adopted [as in Eq. (11)]. All the adjoint (or fictitious) loads (in V, or S or at $t=t_f$) come from the prescribed functional forms of the f, g, and h integrands of I [Eq. (11)].

Material Derivate of I

If for a current shape configuration of the structure (for example, during an iterative solution scheme usually adopted in shape optimization problems) the primary and adjoint problems are satisfied exactly, the MD of I can directly be found from Eq. (25) as follows:

$$\partial_M I=\int_\theta\int_V u_i^*F_i'\,\mathrm{d}V\,\mathrm{d}t$$

$$+\int_\theta\int_S[f-T_{ij}u_{i,j}^*+F_iu_i^*+\rho\dot{u}_i\dot{u}_i^*-D_i\phi_i^*]$$

$$+(g+T_iu_i^*+D\phi^*),_n+H(g+T_iu_i^*+D\phi^*)]V_n\,\mathrm{d}S\,\mathrm{d}t$$

$$+\int_\theta\int_{S_u}\left(\frac{\partial g}{\partial u_i}-T_i^*\right)(\partial_M u_i^0-u_{i,k}V_k)\,\mathrm{d}S\,\mathrm{d}t$$

$$+\int_\theta\int_{S_T}\left(\frac{\partial g}{\partial T_i}+u_i^*\right)(\partial_M T_i^0-T_{i,k}V_k)\,\mathrm{d}S\,\mathrm{d}t$$

$$+\int_\theta\int_{S_\phi}\left(\frac{\partial g}{\partial\phi}-D^*\right)(\partial_M\phi^0-\phi,_kV_k)\,\mathrm{d}S\,\mathrm{d}t$$

$$+\int_\theta\int_{S_D}\left(\frac{\partial g}{\partial D}+\phi^*\right)(\partial_M D^0-D,_kV_k)\,\mathrm{d}S\,\mathrm{d}t$$

$$+\int_\theta\oint_\Gamma[\![g+T_iu_i^*+D\phi^*]\!]\,V_\mu\,\mathrm{d}\Gamma\,\mathrm{d}t$$

$$+\int_V\Big[\rho u_i^*(\partial_M\dot{u}_{i0}-\dot{u}_{i,k}V_k)$$

$$-\left(\frac{\partial h}{\partial u_i}+\rho\dot{u}_i^*\right)(\partial_M u_{i0}-u_{i,k}V_k)\Big]_{t=0}\mathrm{d}V$$

$$+\int_S[h-\rho\dot{u}_iu_i^*]_\theta\,V_n\,\mathrm{d}S \tag{42}$$

Equation (42) thus constitutes the desired shape sensitivity expression of the general performance criterion I for the piezoelectric structure. It is well known that shape derivative calculations are subject to severe limitations for problems involving boundary and shape discontinuities. In the present formulation, however, such jumps are adequately accounted for by the seventh integral on the right-hand side of Eq. (42). It is seen from this equation that in order to evaluate $\partial_M I$, i.e., the total variation of I, the values of the primary and adjoint variables, along with the distribution of the deformation velocity V_i, must be introduced into the equation. For each shape variation of the structure, the primary and adjoint problems must therefore be solved corresponding to the current shape configuration. The similarity in the mathematical forms of the primary and adjoint problems can, however, be used to advantage in numerical evaluations of $\partial_M I$.

Example Problem

A one-dimensional example problem is now used to illustrate the validity of the SSA results obtained for general three-dimensional piezoelectric solids. The one-dimensional case is attractive because it allows comparisons with exact analytical sensitivity results. It has, however, some limitations as a test problem, especially for checking terms involving boundary shape or load discontinuities. Considering a piezoelectric structure of length L, the field equations for the primary problem are given as follows:

$$0<x<L: \quad T_x + F = \rho\ddot{u} \tag{43}$$

$$D_x = 0 \tag{44}$$

where the subscript x denotes differentiation with respect to x, similar notation (without indices) has been adopted for the scalar quantities as in the three-dimensional case, and the body force F is given as

$$F = aC \sin\xi_0 x \tag{45}$$

where a is a constant, C is the elasticity constant of the piezoelectric structure, and $\zeta_0 = (\pi/2L)$. The one-dimensional constitutive equations reduce to the following:

$$T = Cu_x + e\phi_x \tag{46}$$

$$D = eu_x + \epsilon\phi_x \tag{47}$$

Homogeneous boundary and initial conditions are taken in order to simplify the solution, as follows:

$$\text{at } x = 0: \quad u = \phi = 0 \tag{48}$$

$$\text{at } x = L: \quad T = D = 0 \tag{49}$$

$$\text{at } t = 0: \quad u = \dot{u} = 0 \tag{50}$$

With the simple form adopted for the body force F as in Eq. (45), and homogeneous boundary and initial conditions, the solution to the primary problem for the piezoelectric structure is given in terms of the fundamental mode solution in the following form:

$$u(x,t) = (aC/\rho w_0^2)(1 - \cos w_0 t) \sin\zeta_0 x \tag{51}$$

$$\phi(x,t) = (e/\epsilon)u(x,t) \tag{52}$$

where u and ϕ are the one-dimensional elastic and electric displacements; and

$$w_0 = c_0\zeta_0$$

where c_0 is the piezoelectric displacement wave speed given by

$$c_0^2 = 1/\rho[C + (e^2/\epsilon)]$$

A performance criterion I, defined by

$$I = \frac{1}{2}\int_0^{t_f}\int_0^L u^2 \, dx \, dt \tag{53}$$

can be evaluated explicitly in terms of L as

$$I(L) = \frac{a^2C^2L}{16\rho^2 w_0^5}(6w_0t_f - 8\sin w_0t_f + \sin 2w_0t_f) \tag{54}$$

by substituting Eq. (51) into Eq. (53). A direct differentiation of Eq. (54) with respect to L can then be performed so that

$$\frac{dI}{dL} = \frac{a^2C^2}{8\rho^2 w_0^5}[w_0t_f(15 + 4\cos w_0t_f - \cos 2w_0t_f) + 3(\sin 2w_0t_f - 8\sin w_0t_f)] \tag{55}$$

Instead of the direct evaluation and differentiation of I, the general SSA results of the previous section can be utilized to find the total variation of I with respect to L without an explicit evaluation of the performance criterion. Hence, the adjoint problem corresponding to I satisfies the following equations:

$$0<x<L: \quad T_x^* + u = \rho\ddot{u}^* \tag{56}$$

$$D_x^* = 0 \tag{57}$$

$$\text{at } x = D: \quad u^* = \phi^* = 0 \tag{58}$$

$$\text{at } x = L: \quad T^* = D^* = 0 \tag{59}$$

$$\text{at } t = t_f: \quad u^* = \dot{u}^* = 0 \tag{60}$$

where the adjoint constitutive equations are given by

$$T^* = Cu_x^* + e\phi_x^* \tag{61}$$

$$D^* = eu_x^* - \epsilon\phi_x^* \tag{62}$$

Solutions to the adjoint displacements are then simply stated as

$$u^*(x,t) = \frac{aC}{2\rho^2 w_0^4}\{w_0(t_f - t)\sin w_0 t - \sin w_0t_f \sin w_0(t_f - t) + 2[1 - \cos w_0(t_f - t)]\}\sin\zeta_0 x \tag{63}$$

$$\phi^*(x,t) = (e/\epsilon)u^*(x,t) \tag{64}$$

By reducing the general SSA expression [Eq. (42)], it is found that the MD of I [Eq. (51)], is given as follows:

$$\partial_M I = \int_0^{t_f}\int_0^L u^*F' \, dx \, dt + \int_0^{t_f}\left(\frac{1}{2}u^2 + Fu^* + \rho\dot{u}\dot{u}^*\right)_{x=L} V_n \, dt \tag{65}$$

where the partial derivative (with respect to τ) of F can be evaluated as

$$F' = \frac{\partial F}{\partial L}\partial L = -\frac{2aC\zeta_0^2 x}{\pi}\cos\zeta_0 x \partial L \tag{66}$$

Substituting the primary and adjoint solutions into Eq. (65) and taking $V_n = \partial L$ at $x = L$ yield the total variation of I with respect to L in the following form:

$$\partial_M I = \frac{a^2C^2}{8\rho^2 w_0^5}[w_0t_f(15 + 4\cos w_0t_f - \cos 2w_0t_f) + 3(\sin 2w_0t_f - 8\sin w_0t_f)]\partial L \tag{67}$$

which is exactly the same sensitivity expression as has been found previously by the direct evaluation and differentiation of I, as shown in Eq. (55).

Conclusions

The adjoint variable methods of optimization and the MD formulation of shape variations of continua are very effective tools in deriving shape sensitivity expressions without a need for any variational statements regarding the physical phenomenon at hand. It has been demonstrated that coupled fields, as in the case of piezoelectric structures with mechanical and electric effects, under dynamic response can be accommodated easily. Although the first-order sensitivity expressions have been derived systematically by following a step-by-step procedure, higher order sensitivities can also be evaluated by an extension of the procedure. The derived sensitivity expressions, which were checked by an analytical example problem, can be used efficiently in practical shape optimization problems, where discretizations of the primary and adjoint problems might also be required for numerical solution purposes.

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